Soit $n \in \mathbb{N}$, supposons que $0 < U_n < V_n$ montrons que : $0 < U_{n+1} < V_{n+1}$

On a : $U_{n+1} = \dfrac{2U_nV_n}{U_n+V_n} > 0$

$$V_{n+1} = \frac{U_n+V_n}{2} > 0 \quad \text{car} \quad \begin{cases} U_n = 0 \\ V_n = 0 \end{cases}$$

- $U_{n+1} - V_{n+1} = \dfrac{2U_nV_n}{U_n+V_n} - \dfrac{U_n+V_n}{2} = \dfrac{4U_nV_n-(U_n+V_n)^2}{2(U_n+V_n)}$

$$= \frac{4U_nV_n-U_n^2-2U_nV_n-V_n^2}{2(U_n+V_n)} = \frac{-(U_n-V_n)^2}{2(U_n+V_n)} < 0$$

D'où : $0 < U_{n+1} < V_{n+1}$

Conclusion : $\forall n \in \mathbb{N} : 0 < U_n < V_n$

2°/

a) $V_{n+1} - U_{n+1} - \dfrac{1}{2}(V_n - U_n) = \dfrac{(U_n-V_n)^2}{2(U_n+V_n)} - \dfrac{1}{2}(V_n-U_n)$

$$= \frac{(U_nV_n)^2-(V_n+U_n)(V_n+U_n)}{2(U_n+V_n)} = \frac{U_n^2-2U_nV_n+V_n^2-V_n^2+U_n^2}{2(U_n+V_n)}$$

$$= \frac{2U_n(U_n-V_n)}{2(U_n+V_n)} = \frac{U_n(U_n-V_n)}{U_n+V_n} < 0 \quad \text{car :} \quad 0 < U_n < V_n$$

D'où : $V_{n+1} - U_{n+1} \leq \dfrac{1}{2}(V_n - U_n)$

b)

On a : $V_0 - U_0 \leq \left(\dfrac{1}{2}\right)^0$ vraie car : $U_0 = 1$; $V_0 = 2$

Soit $n \in \mathbb{N}$, supposons que : $V_n - U_n \leq \left(\dfrac{1}{2}\right)^n$

On a : $V_{n+1} - U_{n+1} \leq (\dfrac{1}{2}(V_n - U_n) \leq \dfrac{1}{2}\left(\dfrac{1}{2}\right)^n \leq \left(\dfrac{1}{2}\right)^{n+1}$

Conclusion : pour tout $n \in \mathbb{N}$: $V_n - U_n \leq \left(\dfrac{1}{2}\right)^n$